Advanced Studies in Contemporary Mathematics
34 (2024), No. 2, pp. 117 - 121

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2024.34.2.117

ON SHUKLA COHOMOLOGY OF ASSOCIATIVE ALGEBRA BUNDLES

RANJITHA KUMAR, TAEKYUN KIM, AND P. SIVA KOTA REDDY

Abstract. We define Shukla cohomology for associative algebra bundles and we prove the Wedderburn decomposition theorem.

2020 Mathematics Subject Classification. 16DXX, 55RXX.

Keywords and Phrases: Shukla cohomology, Associative algebra bundles, Wedderburn decomposition theorem, Vector bundle.

1. Introduction

The classical "Wedderburn Principal Theorem", defines a finite-dimensional Algebra A over perfect field F as the vector space direct sum of its radical ideal J and sub-algebra S : $A = S \oplus J$. Hochschild presented a cohomological proof for this theorem. The proof reduces to a case for $J^2 = 0$. Linear right inverse of the map $A \to A/J$ is s. In Hochschild cohomology theory, the $s(xy) - s(x)s(y)$ function results in J-valued 2-co-cycle. Because A/J defines a separable F-algebra with vanishing cohomology groups with positive dimension, hence resulting in a map: "$g : A/J \to J$ for $s(xy) - s(x)g(y) - g(xy) + g(x)s(y)$". Therefore, $\psi = s + g$ is an algebra homomorphism which is the right inverse of $A \to A/J$. Also, $A = S \oplus J$ is satisfied if S is the $\psi(A/J)$ sub-algebra. If algebra A is over the general commutative ring K, it is possible that s, a linear-right inverse does not exists [4].

Here we define Shukla cohomology for associative algebra bundles and prove the Wedderburn decomposition theorem.

2. Shukla Cohomology for Associative Algebra Bundles

Let $\xi = \bigcup_{x \in X} \xi_x$ be an associative algebra bundle with unity and $(V, d) = \bigcup_{x \in X} (V_x, d_x)$, where (V_x, d_x) is a differential graded algebra of ξ_x [4]. Then we call (V, d) as graded algebra bundle of ξ. A morphism $\varepsilon : V \to \xi$ is called an augmentation of (V, d) if $\varepsilon_x : V_x \to \xi_x$ is an augmentation if (V_x, d_x). Then (V, d) is free resolution of ξ with exact sequence

$$\cdots \longrightarrow V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \cdots V_0 \xrightarrow{d_0 = \varepsilon} \xi \longrightarrow 0.$$

Define V_n inductively on $\ker d_{n-1}$ elements which are non-trivial. Consider the product of graded tensor $V \otimes \cdots \otimes V = V^{\otimes r}$($r$ times V) with grading $\left(V^{\otimes r}\right)_s$ and natural differential δ_d as the alternating sums of d differential

[1]Corresponding author: tkkim@kw.ac.kr

applied for every entry on one time, further the grade achieved by adding homogeneous element grades in a tensor element. Co-chain groups are defined as, $C^{p,q} = \mathrm{Hom}_k\left((V^{\otimes p})_q, \eta\right)$ of the bi-complex $C^{**}(V \to \xi)$ with vertical differential $\delta_d : C^{p,q} \to C^{p,q+1}$ and horizontal differential $\delta_b : C^{p,q} \to C^{p+1.q}$ defined as follows:

$$\delta_d g\left(a_1, \ldots, a_p\right) = -\sum_{i=1}^{p}(-1)^{e_{i-1}} g\left(a_1, \ldots, da_i, \ldots, a_p\right),$$

$(g \in C^{p,q}, e_i = i + |a_1| + \cdots + |a_i|)$,

$$\begin{aligned}\delta_b g\left(a_1, \ldots, a_{p+1}\right) =& \varepsilon\left(a_1\right) g\left(a_2, \ldots, a_{p+1}\right) \\ &+ \sum_{i=1}^{p}(-1)^{e_t} g\left(a_1, \ldots, a_i a_{i+1}, \ldots, a_{p+1}\right) \\ &+ (-1)^{q+p+1} g\left(a_1, \ldots, a_p\right) \varepsilon\left(a_{p+1}\right).\end{aligned}$$

Shukla cohomology of ξ groups with η values are defined as the cohomology groups of $\left(C^n = \sum_{p+q=n} \oplus C^{p,q}, \delta = \delta_b + \delta_d\right)$ cochain complex, that is $C^{**}(\xi)$, the total complex. These cohomology groups are denoted by $HS^n(\xi, \eta)$; so that,

$$HS^n(\xi, \eta) = H^n\left(C^*, \delta\right).$$

Theorem 2.1. *If J is a nilpotent ideal bundle of an associative algebra bundle ξ such that second Shukla cohomology groups of ξ/J is zero, $HS^2(\xi/J, \eta) = 0$ for every ξ/J-bi-module bundle η. Then there is a sub-algebra bundle $\mathbb{S}$ in ξ such that $\xi = \mathbb{S} \oplus J$.*

Proof. Assume J is the non-trivial ideal bundle which satisfies the condition $J^2 = 0$. Denote ξ/J by B; i, the inclution map of J in ξ; canonical projection $\pi : \xi \to B$. Then we have an extension of B by J

$$0 \longrightarrow J \xrightarrow{i} \xi \xrightarrow{\pi} B \longrightarrow 0.$$

Hence, there is vector bundle morphism defined by, $\kappa : B \to \xi$ [1]. By defining the right and left actions, we can make J into B-bi-module bundle: Define

$$\rho_1 : B \oplus J \to J,$$

by

$$\rho_1(a, u) = \kappa(a)u,$$

and

$$\rho_2 : J \oplus B \to J,$$

by

$$\rho_2(u, a) = u\kappa(a),$$

for all $a \in B_x, u \in J_x$, then J becomes a B-bi-module bundle.

Let $V = \bigcup_{x \in X} V_x$ and $\varepsilon : V \to B$ the standard construction. Define the morphisms, $f : V_0 \times V_0 \to J$ and $g : V_1 \to J$ by

$$f((u),(v))=\kappa(uv)-\kappa(u)\kappa(v),$$
$$g\left(\left(\sum_{i=1}^{n} r_i\left(u_i\right)\right)\right)=\sum_{i=1}^{n} r_i \kappa\left(u_i\right),$$

where $\sum_{i=1}^{n} r_i u_i=0\,(u, v, u_i \in B_x$ and $r_i \in K)$.

Thus, $\sum_{i=1}^{n} r_i\left(u_i\right) \in \operatorname{ker} \varepsilon$ whose pre-image in homogeneous K-base under $d_1: V_1 \rightarrow V_0$ is $\left(\sum_{i=1}^{n} r_i\left(u_i\right)\right)$. Morphism can be defined with an ordinary linear extension as $f \in \operatorname{Hom}_K\left(V_0 \otimes_K V_0, J\right)$ and $g \in \operatorname{Hom}_K\left(V_1, J\right)$. So the "Shukla 2-co-chain" is denoted by (f, g). Now, (f, g) is also proved as the "Shukla 2-co-cycle".

(1) We have

$$\begin{aligned}\left(\delta_b f\right)((u),(v),(w)) = & u f((v),(w))-f((uv),(w))+f((u),(vw))-f((u),(v)) w \\ = & s(u)[s(vw)-s(v) s(w)]-s(uvw)+s(yv) s(w) \\ & +s(uvw)-s(u) s(vw)-[s(uv)-s(u) s(v)] s(w)=0 .\end{aligned}$$

(2) If $u, v_i \in B$ and $\sum_{i=1}^{n} r_i v_i=0$ then

$$\begin{aligned}&\left(\delta_b+\delta_d\right)(f, g)\left((u),\left(\sum r_i\left(v_i\right)\right)\right) \\ &=u g\left(\left(\sum r_i\left(v_i\right)\right)\right)-g\left(\left(\sum r_i\left(u v_i\right)\right)\right)+\sum r_i f\left((u),\left(v_i\right)\right) \\ &=s(u)\left(\sum r_i s\left(v_i\right)\right)-\sum r_i s\left(u v_i\right)+\sum r_i\left[s\left(u v_i\right)-s(u) s\left(v_i\right)\right] \\ &=0 .\end{aligned}$$

(3) Also:

$$\begin{aligned}&\left(\delta_b+\delta_d\right)(f, g)\left(\left(\sum r_i\left(v_i\right)\right),(u)\right) \\ &=g\left(\left(\sum r_i\left(v_i u\right)\right)\right)-g\left(\left(\sum r_i\left(v_i\right)\right)\right) u-\sum r_i f\left(\left(v_i\right),(u)\right) \\ &=\sum r_i s\left(v_i u\right)\Big)-\left[\sum r_i s\left(v_i\right)\right] s(u)-\sum r_i\left[s\left(v_i u\right)-s\left(v_i\right) s(u)\right] \\ &=0 .\end{aligned}$$

(4) For V_2, the $K-$homogeneous base elements is defined as, $\left(\sum_{i=1}^{m} k_i\left(n_i\right)\right)$, for $\sum_{i=1}^{m} k_i n_i=0$ in V_0 and $n_i=\sum_{j=1}^{m_i} r_{ij}\left(u_{ij}\right)$ such that $\sum_j r_{ij} u_{ij}=0$ in B_x for each i value. Hence,

$$\begin{aligned}\left(\delta_d g\right)\left(\left(\sum k_i\left(n_i\right)\right)\right) & =-g\left(\sum k_i\left(n_i\right)\right)=-\sum_{i=1}^{m} k_i g\left(n_i\right) \\ & =-\sum_i k_i g\left(\left(\sum_i r_{ij}\left(u_{ij}\right)\right)\right)=-\sum_i k_i \sum_j r_{ij} s\left(u_{ij}\right) \\ & =-g\left(\left(\sum_{i,j} k_i r_{ij}\left(u_{ij}\right)\right)\right) \\ & =-g(0)=0 .\end{aligned}$$

Since $\sum k_i n_i$ is the last non-trivial argument.

Linearity implies that the pair (f, g) is the "Shukla 2-co-cycle". According to the definition, $HS^2(B, J) = 0$, indicating that normalised Shukla 1-co-chain exists for, $(f, g) = (\delta_b h, \delta_d h)$, $h \in \mathrm{Hom}_K(V_0, J)$. Considering $\psi : B \to A$ map defined by:

$$\psi(u) = s(u) + h((u))$$

for each $u \in B_x$. Next ψ is defined as the algebra bundle morphism and right inverse of π. Proof is complete when $J^2 = 0$, because we assume that $\mathbb{S} = \psi(B)$. To conclude, it is noted that,

(1) $\pi \circ \psi = Id_B$ as $\pi \circ s = Id_B$; $\pi \circ h = 0$;

(2) ψ is K-linear: For, $\sum_{i=1}^{n} r_i u_i \in B_x$, For,

$$\begin{aligned} s\left(\sum r_i u_i\right) - \sum r_i s(u_i) &= g\left(\left(\left(\sum r_i u_i\right) - \sum r_i (u_i)\right)\right) \\ &= (\delta_d h)\left(\left(\left(\sum r_i u_i\right) - \sum r_i (u_i)\right)\right) \\ &= \sum r_i h((u_i)) - h\left(\left(\sum r_i u_i\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} \psi\left(\sum r_i u_i\right) &= s\left(\sum r_i u_i\right) + h\left(\left(\sum r_i u_i\right)\right) \\ &= \sum r_i s(u_i) + \sum r_i h((u_i)) \\ &= \sum r_i \psi(u_i). \end{aligned}$$

(3) ψ is multiplicative:

$$\begin{aligned} \psi(u)\psi(v) &= [s(u) + h((u))][s(v) + h((v))] \\ &= s(u)s(v) + h((u))v + uh((v)) \\ &= s(uv) - f((u), (v)) + \delta_b h((u), (v)) + h((uv)) \\ &= s(uv) + h((uv)) = \psi(uv). \end{aligned}$$

By the standard "Hochschild Induction Argument" on nilpotency degree n of ideal bundle J is used to wrap up the proof. First, observe that our argument for $n = 2$ causes $0 \to J/J^2 \to \xi/J^2 \to B \to 0$ to split. When this happens, a sub-algebra bundle of ξ called ζ exists, and $0 \to J^2 \to \zeta \to B \to 0$ is the exact sequence that also splits according to the "Induction hypothesis". Consider $\mathbb{S}$ to be the image of B in ζ under splitting morphism. Then $\mathbb{S}$ satisfies $A = \mathbb{S} \oplus J$, which completes the proof.

□

References

[1] M. F. Atiyah, *K-theory*, W. A. Benjamin, Inc., New York-Amsterdam, (1967).

[2] C. Chidambara and B. S. Kiranagi, On cohomology of associative algebra bundles, *J. Ramanujan Math. Soc.*, 9(1) (1994), 1–12.

[3] P. J. Hilton and U. Stammbach, *A course in homological algebra*, Grad. Texts Math., 4 (1997), New York, NY: Springer.

[4] L. Kadison, The Wedderburn principal theorem and Shukla cohomology, *J. Pure Appl. Algebra*, 102(1) (1995), 49–60.

[5] B. S. Kiranagi and R. Rajendra, Revisiting Hochschild cohomology for algebra bundles, *J. Algebra Appl.*, 7(6) (2008), 685–715.

Department of Mathematics, School of Advanced Sciences and Languages (SASL), VIT Bhopal University, Kothrikalan, Sehore, Madhya Pradesh-466 114, INDIA.

E-mail address: ranju286math@gmail.com, ranjithakumar@vitbhopal.ac.in

Department of Mathematics, Kwangwoon University, Seoul-139-701, REPUBLIC OF KOREA.

E-mail address: tkkim@kw.ac.kr

Department of Mathematics, JSS Science and Technology University, Mysuru-570 006, INDIA.

E-mail address: pskreddy@jssstuniv.in; pskreddy@sjce.ac.in